5.16 – EXERCÍCIO – pg. 239

1. Determinar o polinômio de Taylor de ordem n, no ponto c dado, das seguintes funções:

a) $f(x) = e^{x/2}$; $c = 0$ e 1; $n = 5$

$$f'(x) = \frac{e^{\frac{x}{2}}}{2}$$

$$f''(x) = \frac{e^{\frac{x}{2}}}{2.2} = \frac{e^{\frac{x}{2}}}{4}$$

$$f'''(x) = \frac{e^{\frac{x}{2}}}{8}$$

$$f^{IV}(x) = \frac{e^{\frac{x}{2}}}{16}$$

$$f^{V}(x) = \frac{e^{\frac{x}{2}}}{32}$$

No ponto $c = 0$:

$$\begin{aligned} P_5(x) &= 1 + \frac{1}{2}x + \frac{1}{4.2!}x^2 + \frac{1}{8.3!}x^3 + \frac{1}{16.4!}x^4 + \frac{1}{32.5!}x^5 \\ &= 1 + \frac{1}{2}x + \frac{1}{8}x^2 + \frac{1}{48}x^3 + \frac{1}{384}x^4 + \frac{1}{3840}x^5 \end{aligned}$$

No ponto $c = 1$:

$$\begin{aligned} P_5(x) &= e^{\frac{1}{2}} + \frac{1}{2}e^{\frac{1}{2}}(x-1) + \frac{1}{4.2!}e^{\frac{1}{2}}(x-1)^2 + \frac{1}{8.3!}e^{\frac{1}{2}}(x-1)^3 + \frac{1}{16.4!}e^{\frac{1}{2}}(x-1)^4 + \frac{1}{32.5!}e^{\frac{1}{2}}(x-1)^5 \\ &= e^{\frac{1}{2}}\left(1 + \frac{1}{2}(x-1) + \frac{1}{8}(x-1)^2 + \frac{1}{48}(x-1)^3 + \frac{1}{384}(x-1)^4 + \frac{1}{3840}(x-1)^5\right) \end{aligned}$$

b) $f(x) = e^{-x}$; $c = -1$ e 2; $n = 4$

$$f'(x) = -e^{-x}$$

$$f''(x) = e^{-x}$$

$$f'''(x) = -e^{-x}$$

$$f^{IV}(x) = e^{-x}$$

$$f^{V}(x) = -e^{-x}$$

Para $c=-1$:

$$P_4(x)=e+(-e^1)\,(x+1)+\frac{e^1}{2!}(x+1)^2-\frac{e^1}{3!}(x+1)^3+\frac{e^1}{4!}(x+1)^4$$
$$=e-e\,(x+1)+\frac{e}{2}(x+1)^2-\frac{e}{6}(x+1)^3+\frac{e}{24}(x+1)^4$$
$$=e\left(1-(x+1)+\frac{(x+1)^2}{2}-\frac{(x+1)^3}{6}+\frac{(x+1)^4}{24}\right)$$

Para $c=2$:

$$P_4(x)=e^{-2}-e^{-2}(x-2)+\frac{e^{-2}}{2!}(x-2)^2-\frac{e^{-2}}{3!}(x-2)^3+\frac{e^{-2}}{4!}(x-2)^4$$
$$=e^{-2}\left(1-(x-2)+\frac{(x-2)^2}{2!}-\frac{(x-2)^3}{3!}+\frac{(x-2)^4}{4!}\right)$$

c) $f(x)=\ln(1-x);\, c=0 \text{ e } 1/2;\, n=4$

$$f'(x)=\frac{-1}{1-x}$$
$$f''(x)=\frac{(1-x).0+(-1)}{(1-x)^2}=\frac{-1}{(1-x)^2}$$
$$f'''(x)=\frac{2(1-x).(-1)}{(1-x)^4}=\frac{-2}{(1-x)^3}$$
$$f^{IV}(x)=\frac{2.3(1-x)^2.(-1)}{(1-x)^6}=\frac{-6}{(1-x)^4}$$
$$f^{V}(x)=\frac{6.4(1-x)^3.(-1)}{(1-x)^8}=\frac{-24}{(1-x)^5}$$

Para $c=0$:

$$P_4(x)=0+\frac{-1}{1}x-\frac{1}{2!}x^2-\frac{2}{3!}x^3-\frac{6}{4!}x^4$$

Para $c=1/2$:

$$P_4(x)=-\ln 2-2\left(x-\frac{1}{2}\right)-\frac{4}{2!}\left(x-\frac{1}{2}\right)^2-\frac{16}{3!}\left(x-\frac{1}{2}\right)^3-\frac{96}{4!}\left(x-\frac{1}{2}\right)^4$$

d) $f(x)=sen\ x;\, c=\pi/2;\, n=8$

$$f'(x) = \cos x = f^{V}$$
$$f''(x) = -sen\, x = f^{VI}$$
$$f'''(x) = -\cos x = f^{VII}$$
$$f^{IV}(x) = sen\, x = f^{VIII}$$

$$P_8(x) = 1 + \frac{(-1)}{2!}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{2}\right)^4 - \frac{1}{6!}\left(x - \frac{\pi}{2}\right)^6 + \frac{1}{8!}\left(x - \frac{\pi}{2}\right)^8$$

e) $f(x) = \cos 2x;\ c = 0 \text{ e } \pi/2;\ n = 6$

$$f^{I}(x) = -2sen\, 2x$$
$$f^{II}(x) = -4\cos 2x$$
$$f^{III}(x) = 8sen\, 2x$$
$$f^{IV}(x) = 16\cos 2x$$
$$f^{V}(x) = -32sen\, 2x$$
$$f^{VI}(x) = -64\cos 2x$$

Para $c = 0$:

$$\begin{aligned} P_6(x) &= 1 - \frac{4}{2!}x^2 + \frac{16}{4!}x^4 - \frac{64}{6!}x^6 \\ &= 1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 \end{aligned}$$

Para $c = \frac{\pi}{2}$:

$$\begin{aligned} P_6(x) &= -1 + \frac{4}{2!}\left(x - \frac{\pi}{2}\right)^2 - \frac{16}{4!}\left(x - \frac{\pi}{2}\right)^4 + \frac{64}{6!}\left(x - \frac{\pi}{2}\right)^6 \\ &= -1 + 2\left(x - \frac{\pi}{2}\right)^2 - \frac{2}{3}\left(x - \frac{\pi}{2}\right)^4 + \frac{4}{45}\left(x - \frac{\pi}{2}\right)^6 \end{aligned}$$

f) $f(x) = \dfrac{1}{1+x};\ c = 0 \text{ e } 1;\ n = 4$

$$f^{I}(x)=\frac{-1}{(1+x)^2}$$

$$f^{II}(x)=\frac{2}{(1+x)^3}$$

$$f^{III}(x)=\frac{-6}{(1+x)^4}$$

$$f^{IV}(x)=\frac{24}{(1+x)^5}$$

Para $c=0$:

$$\begin{aligned}P_4(x)&=1-x+\frac{2}{2!}x^2-\frac{6}{3!}x^3+\frac{24}{4!}x^4\\&=1-x+x^2-x^3+x^4\end{aligned}$$

Para $c=1$:

$$\begin{aligned}P_4(x)&=-\frac{1}{2}-\frac{1}{4}(x-1)+\frac{1}{4.2!}(x-1)^2-\frac{3}{8.3!}(x-1)^3+\frac{3}{4.4!}(x-1)^4\\&=-\frac{1}{2}-\frac{1}{4}(x-1)+\frac{1}{8}(x-1)^2-\frac{1}{16}(x-1)^3+\frac{1}{32}(x-1)^4\end{aligned}$$

2. Encontrar o polinômio de Taylor de grau n no ponto c e escrever a função que define o resto na forma de Lagrange, das seguintes funções:

a) $y=\cosh x,\ n=4,\ c=0$

$$y=\cosh x=\frac{e^0+e^{-0}}{2}=1$$

$$y^{I}=senhx=\frac{e^0-e^{-0}}{2}=0$$

$$y^{II}=\cosh x=\frac{e^0+e^{-0}}{2}=1$$

$$y^{III}=senhx=\frac{e^0-e^{-0}}{2}=0$$

$$y^{IV}=\cosh x=\frac{e^0+e^{-0}}{2}=1$$

$$P_4(x) = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4$$

$$P_4(x) = 1 + \frac{x^2}{2} + \frac{x^4}{24}$$

$$R_4(x) = \frac{f^{v}(z)}{5!}(x-0)^5$$

$$R_4(x) = \frac{senhz}{5!}x^5 \quad \text{onde} \quad z \text{ é um n° entre 0 e } x.$$

b) $y = tgx,\ n = 3,\ c = \pi$

$$y = tgx \Rightarrow y(\pi) = 0$$

$$y' = \sec^2 x \Rightarrow y'(\pi) = 1$$

$$y'' = 2\sec^2 x.tgx \Rightarrow y''(\pi) = 0$$

$$y''' = 2\sec^2 x.\sec^2 x + tgx.4\sec^2 x.tgx \Rightarrow y'''(\pi) = 2$$

$$y^{IV} = 8\sec^3 x.\sec x.tgx + 4\sec^2 x.\sec^2 x.(-2tgx) + tg^2 x.8\sec^2 x.tgx$$

$$P_3(x) = 1(x-\pi) + \frac{2(x-\pi)^3}{3!}$$

$$= x - \pi + \frac{(x-\pi)^3}{3}$$

$$R_3(x) = \frac{16\sec^4 z.tgz + 8\sec^2 z.tg^3 z}{4!}(x-\pi)^4$$

c) $y = \sqrt{x};\ n = 3;\ c = 1$

$$y = x^{1/2} \Rightarrow y(1) = 1$$

$$y' = \frac{1}{2}x^{-1/2} \Rightarrow y'(1) = \frac{1}{2}$$

$$y'' = -\frac{1}{4}x^{-3/2} \Rightarrow y''(1) = -\frac{1}{4}$$

$$y''' = \frac{3}{8}x^{-5/2} \Rightarrow y'''(1) = \frac{3}{8}$$

$$y^{IV} = -\frac{15}{16}x^{-7/2}$$

$$P_3(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{4}\cdot\frac{1}{2!}(x-1)^2 + \frac{3}{8}\cdot\frac{1}{3!}(x-1)^3$$
$$= 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3$$
$$R_3(x) = \frac{-15}{16z^3\sqrt{z}}\cdot\frac{1}{24}(x-1)^4$$

d) $y = e^{-x^2}$; n = 4; c = 0.

$$y = e^{-x^2}$$
$$y^{I} = -e^{-x^2}\ 2x$$
$$y^{II} = -e^{-x^2}\ 2 + 2x.-e^{-x^2}\ (-2x)$$
$$= -2e^{-x^2} + 4x^2\ e^{-x^2}$$
$$y^{III} = -2e^{-x^2}(-2x) + 4x^2e^{-x^2}(-2x) + e^{-x^2}\ .8x$$
$$y^{IV} = -48x^2e^{-x^2} + 16x^4e^{-x^2} + 12e^{-x^2}$$
$$y^{V} = 160x^3e^{-x^2} - 120xe^{-x^2} - 32x^5e^{-x^2}$$

$$y(0) = 1$$
$$y^{I}(0) = 0$$
$$y^{II}(0) = -2$$
$$y^{III}(0) = 0$$
$$y^{IV}(0) = 12$$

$$P_4(x) = 1 - \frac{2}{2!}x^2 + \frac{12}{4!}x^4$$
$$= 1 - x^2 + \frac{x^4}{2}$$
$$R_4(x) = \frac{e^{-z^2}}{5!}\left(160z^3 - 120z - 32z^5\right)x^5$$

3. Usando o resultado encontrado no exercício 1, item (c), com $c = 0$, determinar um valor aproximado para $\ln 0{,}5$. Fazer uma estimativa para o erro.

$$\ln 0{,}5 = P_4(0{,}5) + R_4(0{,}5)$$

$$P_4(x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4$$

$$R_4(x) = \frac{-24}{(1-z)^5} \cdot \frac{1}{5!}x^5$$

$$= \frac{-x^5}{5(1-z)^5}$$

$$\Rightarrow \ln 0{,}5 = -0{,}5 - \frac{1}{2}(0{,}5)^2 - \frac{1}{3}(0{,}5)^3 - \frac{1}{4}(0{,}5)^4 + \frac{-(0{,}5)^5}{5(1-z)^5}$$ onde z é um número entre 0 e 0,5.

$$\ln 0{,}5 = -0{,}5 - 0{,}125 - 0{,}041666 - 0{,}015625$$
$$= -0{,}682292$$

$$0 < z < 0{,}5$$
$$-0{,}5 < -z < 0$$
$$0{,}5 < 1 - z < 1$$
$$\frac{1}{0{,}5} > \frac{1}{1-z} > 1$$
$$\frac{1}{0{,}5^5} > \frac{1}{(1-z)^5} > 1$$

$$\left|R_4(0{,}5)\right| = \left|\frac{-0{,}00625}{(1-z)^5}\right| < \frac{0{,}00625}{0{,}5^5} = 0{,}2$$

4. Determinar o polinômio de Taylor de grau 6 da função *f(x)* = 1 + cos *x* no ponto $c = \pi$. Usar este polinômio para determinar um valor aproximado para $\cos 5\pi/6$. Fazer uma estimativa para o erro.

$f(x) = 1 + \cos x \Rightarrow f(\pi) = 0$

$f^{I}(x) = 0 - senx \Rightarrow f'(\pi) = 0$

$f^{II}(x) = -\cos x \Rightarrow f''(\pi) = 1$

$f^{III}(x) = senx \Rightarrow f'''(\pi) = 0$

$f^{IV}(x) = \cos x \Rightarrow f'^{v}(\pi) = -1$

$f^{V}(x) = -senx \Rightarrow f^{v}(\pi) = 0$

$f^{VI}(x) = -\cos x \Rightarrow f^{VI}(\pi) = 1$

$f^{VII}(x) = senx$

$$P_6(x) = \frac{1}{2!}(x-\pi)^2 - \frac{1}{4!}(x-\pi)^4 + \frac{1}{6!}(x-\pi)^6$$
$$= \frac{1}{2}(x-\pi)^2 - \frac{1}{24}(x-\pi)^4 + \frac{1}{720}(x-\pi)^6$$

$$\cos x = f(x) - 1$$

$$\cos x \cong -1 + \frac{1}{2}(x-\pi)^2 - \frac{1}{24}(x-\pi)^4 + \frac{1}{720}(x-\pi)^6$$

$$\cos\left(\frac{5\pi}{6}\right) \cong -1 + \frac{1}{2}\left(\frac{5\pi}{6} - \pi\right)^2 - \frac{1}{24}\left(\frac{5\pi}{6} - \pi\right)^4 + \frac{1}{720}\left(\frac{5\pi}{6} - \pi\right)^6$$
$$= -1 + 0{,}13707 - 0{,}0031317 + 0{,}000028619$$
$$= -0{,}8660331$$

$$R_6(x) = \frac{senz}{7!}(x-\pi)^7$$

$$R_6\left(\frac{5\pi}{6}\right) = \frac{senz}{7!}\left(\frac{5\pi}{6} - \pi\right)^7$$

$$|senz| \leq 1$$

$$\left|R_6\left(\frac{5\pi}{6}\right)\right| \leq \frac{\left(\frac{5\pi}{6} - \pi\right)^7}{7!} \cong 0{,}0000213$$

5. Demonstrar que a diferença entre sen($a + h$) e $sen\, a + h \cos a$ é menor ou igual a $\frac{1}{2}h^{2.}$.

$y = senx$

$y' = \cos x$

$y'' = -senx$

$$senx = sena + \cos a(x-a) + R_1(x)$$
$$sen(a+h) = sena + \cos a(a+h-a) + R_1(a+h)$$
$$sen(a+h) = sena + h\cos a + R_1(a+h)$$

$$R_1(a+h) = \frac{-senz}{2}h^2$$
$$\left|R_1(a+h)\right| = \left|\frac{-senz}{2}h^2\right|$$
$$= \frac{h^2}{2}\left|-senz\right|$$
$$= \frac{h^2}{2}\left|senz\right|$$
$$\le \frac{h^2}{2}$$

6. Um fio delgado, pela ação da gravidade, assume a forma da catenária $y = a\cosh\frac{x}{a}$. Demonstrar que, para valores pequenos de $|x|$, a forma que o fio toma pode ser representada, aproximadamente, pela parábola $y = a + \frac{x^2}{2a}$.

$$y = a\cosh\frac{x}{a} \Rightarrow y(0) = a$$
$$y' = \frac{a}{a}senh\frac{x}{a} = senh\frac{x}{a} \Rightarrow y'(0) = 0$$
$$y'' = \frac{1}{a}\cos h\frac{x}{a} \Rightarrow y''(0) = \frac{1}{a}$$

$$P_2(x) = a + \frac{1}{a}.\frac{1}{2!}x^2$$
$$= a + \frac{x^2}{2a}$$

7. Pesquisar máximos e mínimos das seguintes funções:

a) $f(x) = 2x - 4$

$f'(x) = 2$ ∄ pontos críticos

∄ máximos nem mínimos.

b) $f(x) = 4 - 5x + 6x^2$

$f'(x) = -5 + 12x$

$-5 + 12x = 0$

$12x = 5$

$x = \dfrac{5}{12}$

$f''(x) = 12$

$f''\big|_{5/12} > 0 \Rightarrow x = \dfrac{5}{12}$ é ponto de mínimo

c) $f(x) = (x-4)^{10}$

$f'(x) = 10(x-4)^9$

$10(x-4)^9 = 0 \quad \Rightarrow \quad x = 4$

$f^{II}(x) = 90(x-4)^8$

$f^{III}(x) = 720(x-4)^7$

$f^{IV}(x) = 5040(x-4)^6$

$f^{V} = f^{VI} = f^{VII} = f^{VIII} = f^{9} = 0$

$f^{10} = K > 0 \quad \Rightarrow x = 4$ é ponto de mínimo.

d) $f(x) = 4(x+2)^7$

$f^{I}(x) = 28(x+2)^6$

$f^{II}(x) = k(x+2)^5$

$f^{III}(x) = k_1(x+2)^4$

$f^{IV}(x) = k_2(x+2)^3$

$f^{V}(x) = k_3(x+2)^2$

$f^{VI}(x) = k_4(x+2)$

$f^{VII}(x) = k_5$

$\Rightarrow \quad x = -2$ é ponto de inflexão.

e) $f(x) = x^6 - 2x^4$

$$f(x) = x^6 - 2x^4$$

$$f'(x) = 6x^5 - 8x^3$$

$$6x^5 - 8x^3 = 0$$

$$x^3(6x^2 - 8) = 0$$

$$x_1 = 0 \qquad x_2 = \frac{2}{\sqrt{3}} \qquad x_3 = -\frac{2}{\sqrt{3}}$$

$$f''(x) = 30x^4 - 24x^2$$

$$f'''(x) = 120x^3 - 48x$$

$$f^{IV}(x) = 360x^2 - 48$$

$f^{IV}\big|_0 < 0 \Rightarrow x_1 = 0$ é ponto de máximo

$f''\big|_{\frac{2}{\sqrt{3}}} > 0 \Rightarrow x = \frac{2}{\sqrt{3}}$ é ponto de mínimo.

$f''\big|_{\frac{-2}{\sqrt{3}}} > 0 \Rightarrow x = -\frac{2}{\sqrt{3}}$ é ponto de mínimo.

f) $f(x) = x^5 - \dfrac{125}{3}x^3$

$$f'(x) = 5x^4 - \frac{125}{3}3x^2$$

$$5x^4 - 125x^2 = 0$$

$$x^2(5x^2 - 125) = 0$$

$$5x^2 - 125 = 0$$

$$5x^2 = 125$$

$$x^2 = \frac{125}{5}$$

$$x^2 = 25$$

$$x_1 = 0 \qquad x_2 = 5 \qquad x_3 = -5$$

$$f'' = 20x^3 - 250x$$

$f''\big|_5 > 0 \Rightarrow x = 5$ é ponto de mínimo.

$f''\big|_{-5} = -2500 + 1250 < 0 \Rightarrow x = -5$ é ponto de máximo.

$$f''' = 60x^2 - 250$$

$f'''\big|_0 \neq 0 \Rightarrow x = 0$ é ponto de inflexão.